

## Vectors & Dot Product

**Scalar Quantities:** Quantities such as length, area, volume, temperature, and time, which have magnitude (size), but no direction.

**Vector Quantities:** Quantities that involve both a magnitude and direction, such as velocity, acceleration, and force. These quantities can be represented by **directed line segments** called **vectors**.

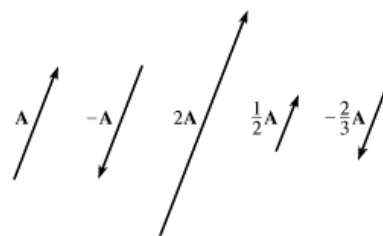
The length of a vector represents the **magnitude** of the vector quantity. The **direction** is indicated by the position of the vector and the arrowhead at one end.

**Notation:**  $\overrightarrow{AB}$  is used to name a vector with **initial point**  $A$  and **terminal point**  $B$ . Vectors may also be denoted by bold letters.  $\overrightarrow{AB}$  can also be written as  $\mathbf{AB}$ . If the initial and terminal points are not specified, vectors can be named by a single uppercase or lowercase letter (eg.  $\vec{b}$ ,  $\vec{B}$ ,  $\mathbf{b}$ , or  $\mathbf{B}$ .) The magnitude of vector  $\mathbf{A}$  is written  $|\mathbf{A}|$ .

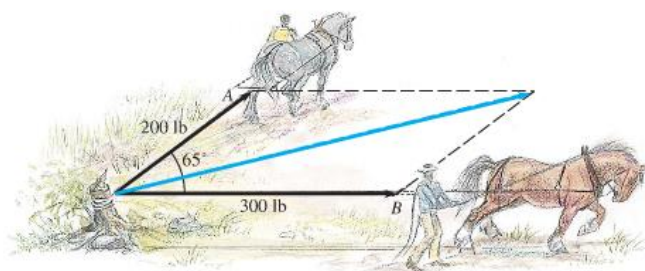
**Equal Vectors:** Vectors with the same magnitude and direction. They do not have to be in the same place.

**Zero Vector:** A vector with no magnitude and no direction. It is denoted by  $\mathbf{0}$ .

**Scalar Multiplication:** For any scalar  $k$  and vector  $\mathbf{A}$ ,  $k\mathbf{A}$  is a vector with magnitude  $|k|$  times the magnitude of  $\mathbf{A}$ . If  $k > 0$ , then the direction of  $k\mathbf{A}$  is the same as the direction of  $\mathbf{A}$ . If  $k < 0$ , the direction of  $k\mathbf{A}$  is opposite to the direction of  $\mathbf{A}$ . If  $k = 0$ , then  $k\mathbf{A} = \mathbf{0}$ .

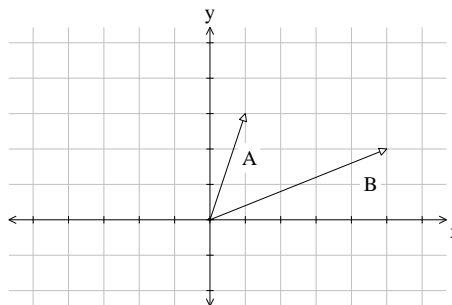


Suppose two draft horses are pulling on a tree stump with forces of 200 pounds and 300 pounds, with an angle of  $65^\circ$  between the forces. If  $\mathbf{A}$  and  $\mathbf{B}$  had the same direction, then there would be a total force of 500 pounds acting on the stump, but the total force is less because of the angle between the forces. By the **parallelogram law**, the force acting along the diagonal of the parallelogram, with a magnitude equal to the length of the diagonal, has the same effect on the stump as the two forces  $\mathbf{A}$  and  $\mathbf{B}$ . The force  $\mathbf{A} + \mathbf{B}$  acting along the diagonal is called the **sum** or **resultant** of  $\mathbf{A}$  and  $\mathbf{B}$ .



**Vector Addition:** To find the resultant or sum  $\mathbf{A} + \mathbf{B}$  of any vectors  $\mathbf{A}$  and  $\mathbf{B}$ , position  $\mathbf{B}$  (without changing its magnitude or direction) so that the initial point of  $\mathbf{B}$  coincides with the terminal point of  $\mathbf{A}$ . The vector that begins at the initial point of  $\mathbf{A}$  and ends at the terminal point of  $\mathbf{B}$  is the vector  $\mathbf{A} + \mathbf{B}$ . For every vector  $\mathbf{A}$ , there is a vector  $-\mathbf{A}$ , with the same magnitude as  $\mathbf{A}$ , but the opposite direction. For any two vectors  $\mathbf{A}$  and  $\mathbf{B}$ ,  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ .

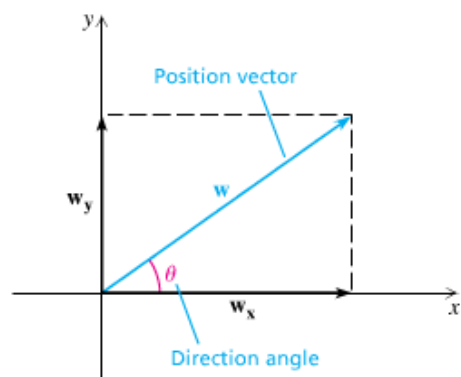
**Example:** Sketch the vectors  $\mathbf{A} + \mathbf{B}$  and  $\mathbf{A} - \mathbf{B}$ .



Any nonzero vector  $\mathbf{w}$  is the sum of a **horizontal component**,  $\mathbf{w}_x$ , and a **vertical component**,  $\mathbf{w}_y$ . If a vector  $\mathbf{w}$  is placed in a rectangular coordinate system so that its initial point is the origin, then  $\mathbf{w}$  is called a **position vector**. The angle  $\theta$  formed by the positive  $x$ -axis and a position vector is the **direction angle** for the position vector.

If the vector  $\mathbf{w}$  has magnitude  $r$ , direction angle  $\theta$ , horizontal component  $\mathbf{w}_x$ , and vertical component  $\mathbf{w}_y$ , then we get

$$\cos \theta = \frac{|\mathbf{w}_x|}{r} \text{ and } \sin \theta = \frac{|\mathbf{w}_y|}{r} \text{ or } |\mathbf{w}_x| = |r \cos \theta| \text{ and } |\mathbf{w}_y| = |r \sin \theta|.$$



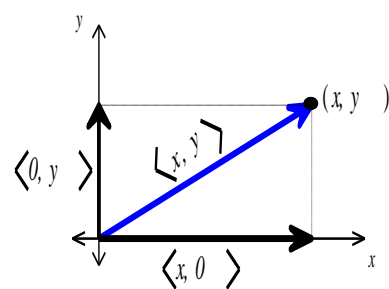
**Examples:** Find the magnitude of the horizontal and vertical components for each vector  $\mathbf{v}$  with the given magnitude and direction angle  $\theta$ . Round to the nearest tenth.

a)  $|\mathbf{v}| = 5.6$ ,  $\theta = 22^\circ$

b)  $|\mathbf{v}| = 445$ ,  $\theta = 211.1^\circ$

**Component Form:** The notation  $\langle x, y \rangle$  is used to define a position vector with terminal point  $(x, y)$ . This is called component form because the horizontal component is  $\langle x, 0 \rangle$  and its vertical component is  $\langle 0, y \rangle$ .

The magnitude of the vector  $\mathbf{v} = \langle x, y \rangle$  is  $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$ . To find the direction angle, use  $\sin \theta = \frac{y}{r}$ ,  $\cos \theta = \frac{x}{r}$ , and  $\tan \theta = \frac{y}{x}$ .



If a vector has magnitude  $r$  and direction angle  $\theta$ , its component form is  $\langle r \cos \theta, r \sin \theta \rangle$ .

**Examples:** Find the magnitude and direction angle of each vector.

a)  $\mathbf{v} = \langle 2, -6 \rangle$

b)  $\mathbf{v} = \langle -3, 2 \rangle$

**Examples:** Find the component form for each vector  $\mathbf{v}$  with the given magnitude and direction angle  $\theta$ .

a)  $|\mathbf{v}| = 12$ ,  $\theta = 45^\circ$

b)  $|\mathbf{v}| = 50$ ,  $\theta = 120^\circ$

If  $\mathbf{A} = \langle a_1, a_2 \rangle$ ,  $\mathbf{B} = \langle b_1, b_2 \rangle$ , and  $k$  is a scalar, then

	1. $k\mathbf{A} = \langle ka_1, ka_2 \rangle$	<b>Scalar Product</b>
<b>Vector Arithmetic:</b>	2. $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$	<b>Vector Sum</b>
	3. $\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$	<b>Vector Difference</b>
	4. $\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2$	<b>Dot Product</b>

**Examples:** Let  $\mathbf{w} = \langle -1, -3 \rangle$  and  $\mathbf{v} = \langle 3, -4 \rangle$ . Perform the operations indicated.

a)  $\mathbf{w} - \mathbf{v}$

b)  $-8\mathbf{v}$

c)  $3\mathbf{w} + 4\mathbf{v}$

d)  $\mathbf{w} \cdot \mathbf{v}$

### The Angle Between Two Vectors:

If  $\mathbf{A}$  and  $\mathbf{B}$  are nonzero vectors and  $\alpha$  is the smallest positive angle between them, then  $\cos \alpha = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$ .

**Examples:** Find the smallest positive angle between the following vectors:

a)  $\langle 1, 3 \rangle$  and  $\langle 5, 2 \rangle$

b)  $\langle -1, 5 \rangle$  and  $\langle 2, 7 \rangle$

**Parallel Vectors:** The vectors  $\mathbf{A}$  and  $\mathbf{B}$  are parallel if and only if  $\mathbf{A} = k\mathbf{B}$  for a nonzero scalar  $k$ .

**Perpendicular Vectors:** The vectors  $\mathbf{A}$  and  $\mathbf{B}$  are perpendicular if and only if  $\mathbf{A} \cdot \mathbf{B} = 0$ .

**Examples:** Determine whether each pair of vectors is parallel, perpendicular, or neither.

a)  $\langle -2, 3 \rangle$  and  $\langle 6, 4 \rangle$

b)  $\langle 2, -5 \rangle$  and  $\langle -4, 10 \rangle$

c)  $\langle 2, 6 \rangle$  and  $\langle 6, 2 \rangle$

The vectors  $\mathbf{i} = \langle 1, 0 \rangle$  and  $\mathbf{j} = \langle 0, 1 \rangle$  are called **unit vectors** because each has magnitude one. For any vector  $\langle a_1, a_2 \rangle$ , we have  $\langle a_1, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$ . The form  $a_1\mathbf{i} + a_2\mathbf{j}$  is called a **linear combination** of the vectors  $\mathbf{i}$  and  $\mathbf{j}$ .

**Examples:** Write each vector as a linear combination of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ .

a)  $\mathbf{A} = \langle 2, 3 \rangle$

b)  $\mathbf{B} = \langle -1, 7 \rangle$

c)  $\mathbf{C} = \langle 0, -9 \rangle$